ADIA

Strategies and Actors in Financial Markets
Microstructure and flow trading for Intermediaries

Amine Raboun, Ph.D
Quantitative Researcher & Developer - Abu Dhabi Investment Authority
Lecturer, Paris Dauphine – PSL University

Outline

1. The role of intermediaries: The position of the middleman
    1. Between different investors
    2. In between investors and infrastructure
    3. Organization inside a brokerage firm
2. The price formation process
    1. Liquidity and Market Efficiency
    2. The emergence of new market structure
3. Focus on execution
    1. Essentials of Microstrucutre
    2. Algorithmic trading
    3. Pre and Post trade analysis, TCA, consultancy
    4. Direct Market Access
4. Conclusion

The role of Intermediaries
The position of middleman

Middleman Role: Issuers <-> Investors

- Corporate Brokerage
  - Investment Banks have senior bankers, working on Corporates' debt)
  - When a company issue shares (or bonds), she needs a *book runner* to be in between her and the investors. Usually, the Broker guarantees a price per share (conditioned by the number of shares sold the first day).
  - It is the same to raise capital on markets (conditions are on volatility around the issuance and market volumes, guaranteed prices are pegged on market VWAPs),
  - it is the same for share buy back programs.
  - Some brokers are mandated by one company to make the market on their stocks.

- Corporate Access
  - Brokers organize meetings (one-to-one or conferences) between issuers (mostly CFOs) and investors
  - Companies have the opportunity to explain their strategy
  - Investors have the opportunity to challenge the accuracy of explanations.

- Analysis (research) -- Brokers issue reports on
  - macroeconomics (strategists)
  - companies and sectors (fundamental analysts)
  - systematic (factorial) portfolios (quantitative analysts).

Middleman Role: Investors <-> Investors

- Block trading (OTC) - ``upstairs trading''
  - Broker's Sales traders (qualitatively) know the portfolio of investors (anonymity)
  - Especially because the Broker's Sales sold them Analysts' ideas
  - Some small brokerage firms are specialized in sectorial small cap liquidity seeking.
  - Can have discretion to find blocks for one week.
- Market Easing – Facilitation
  - Take quantity at a negotiated price
  - by voice for large quantities
  - by electronic means for small quantities (answers at “the touch”).
  - a broker provides liquidity on other asset classes too, if needed.
- Members of exchange – “common trading”
  - Historically brokers were the founders (and only members) of exchanges
  - Now they are clients of exchanges, and no more the first clients (market makers and HFT are more important).
  - In some exchanges, brokers can be privileged provided they pay for.
  - Brokers guarantee best execution, provide Direct Market Access and electronic services (algo trading, TCA and consultancy).

Middleman Role: Investors <-> Infrastructure

- Regulation and market surveillance
  - Brokers are part of the surveillance infrastructure (risk limits)
  - They have to warn national authorities when they detect suspicious behavior
  - They work with exchanges and authorities when asked to
- Clearing, Collateralization and Settlement
  - Brokers dispatch drop-copies of trades to keep track of investor's positions where needed
  - Brokers are connected to CCPs and custodians, thus play the role of middlemen between end investor and infrastructures
- Repo and financing
  - Prime brokers are providing repo, reverse repo and financing

Who pays whom ?

- Issuers pay for corporate brokerage
  - but they come to brokers' conferences for free,
  - *in exchange* they increase their chances to be included into investors' portfolios.
- Investors pay fees for trading
  - They have to trade anyway; thus, they consider they have to pay the fees
  - Before MIFID II, investors used to choose to pay high touch or low touch fees.
  - With high touch they have access to financial analysis (research) on top of electronic trading
  - MIFID II oblige brokers to unbundle execution and research, and to pay hard dollars for reports and meeting.
- Investors vote at Extel Refinitiv and MorningStar surveys.
  - Most Buy-sides are using the Extel ranking as a repartition key to split their flows amongst brokers.
  - MorningStar survey is more quantitative than Extel, but less used.
  - This is a reinforcing ecosystem: the more clients you have the best ranked, thus it is very difficult for newcomers. This is *good for concentration*.
- Investment banks pay to have a better market access and a better access to information.
  - the Brokerage business is overcrowded, nevertheless some Investment banks pay to maintain their brokerage arm alive.

Organization inside a brokerage firm

In a typical organization:

- **Sales:** select and distribute the reports, they are in charge of the global sales relationship
- **Analysts:** write the reports, talk to the CFO of the corporates and to investors 'Portfolio managers.
- **Sales-traders:** take investors orders (by phone) via Dealing Desks and seek liquidity> They select and send flow-oriented reports (often written by quant analysts).They are in charge of upstairs trading , and route the remaining orders to traders or trading algorithms
- **Traders:** mix manual and algorithmic traded orders
- **Dealers:** manage broker’s prop books
- **AES sales-traders:** receive and monitor algo orders sent directly by low touch clients. They are the only one to see the low touch flows. They offer execution consultancy to clients.
- **DMA teams:** are monitorigin DMA orders and operate and help desk.
- **Middle and back offices:** maintain the relationship with investors middle and back offices and with CCPS

The Price Formation Process
Liquidity and Market efficiency

Liquidity contributes to price dynamics

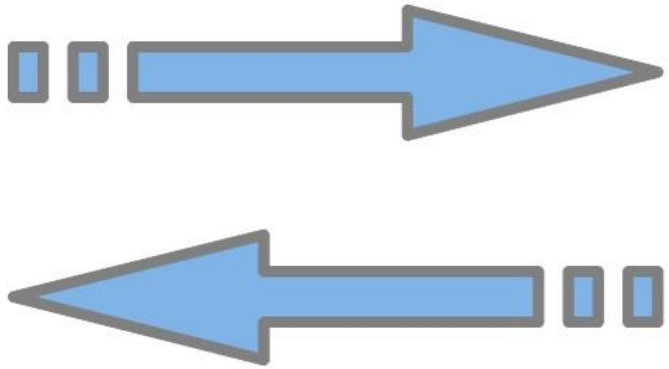
The dynamic of the Price Fromation (or Price Discovery) Process is twofold:

Forward dynamics

The balance between offer and demand at t move at t+1

Backward dynamics

The comparison between current prices and future prices expectations triggers new offers and demands



It simultaneously creates (endogenous) information and conveys (exogenous) information.

A liquid market is a mixed market

A **trading facility has two main goals:**

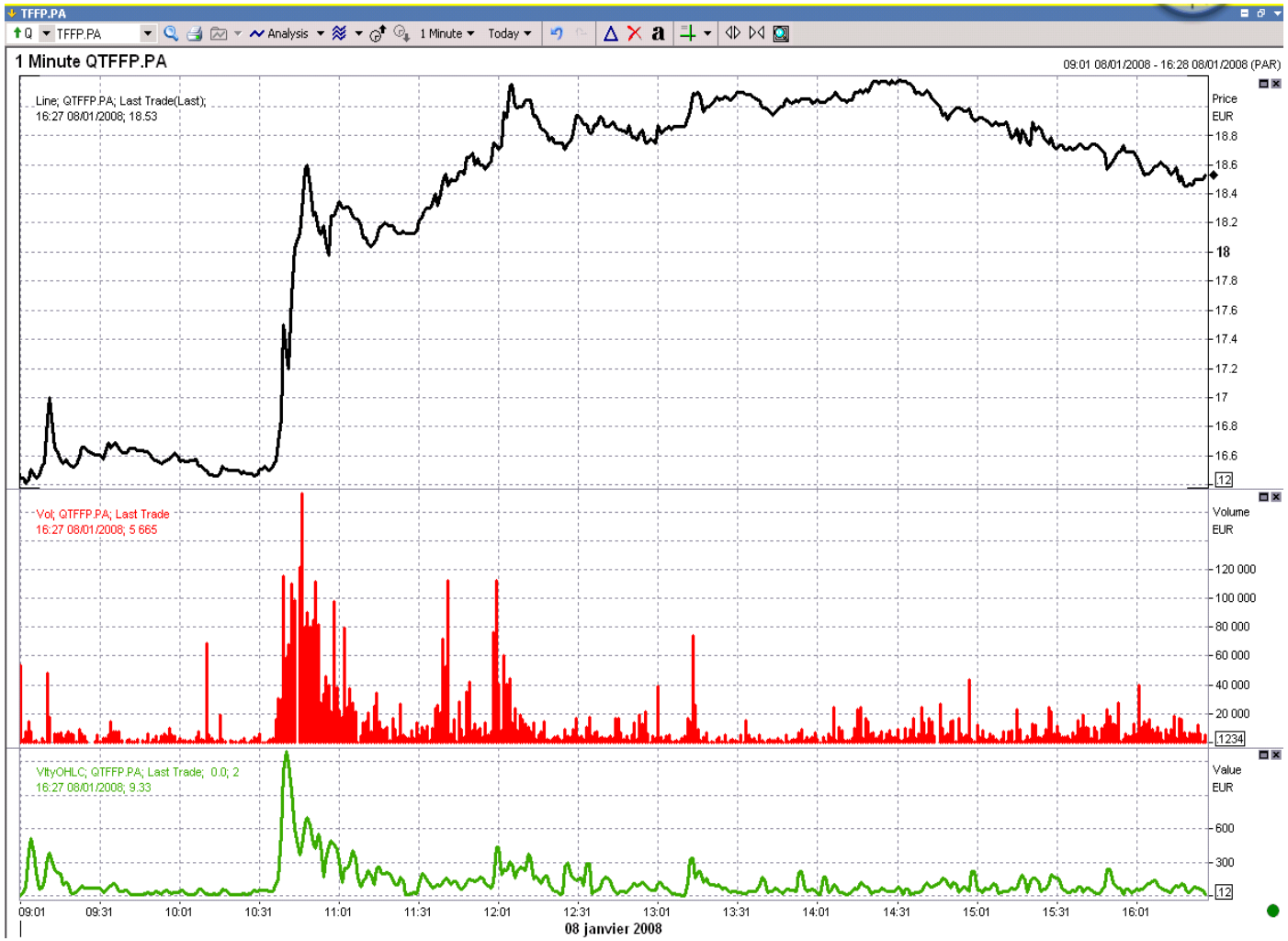
- Achieve an equilibrium price that matches the offer and demand. The more buyers, the higher the price and the more sellers the lower the price. This is the source of market pressure
- Disseminate prices that digested most of the available information concerning the valuation of assets

The **synchronization of sellers and buyers** is the main component of illiquidity (and thus market impact):

- When an external news is disclosed: the synchronization is immediate;
- When information affect differently investment time horizons: the impact is smoother;
- A market with different investment philosophies and different time horizons is more liquid.

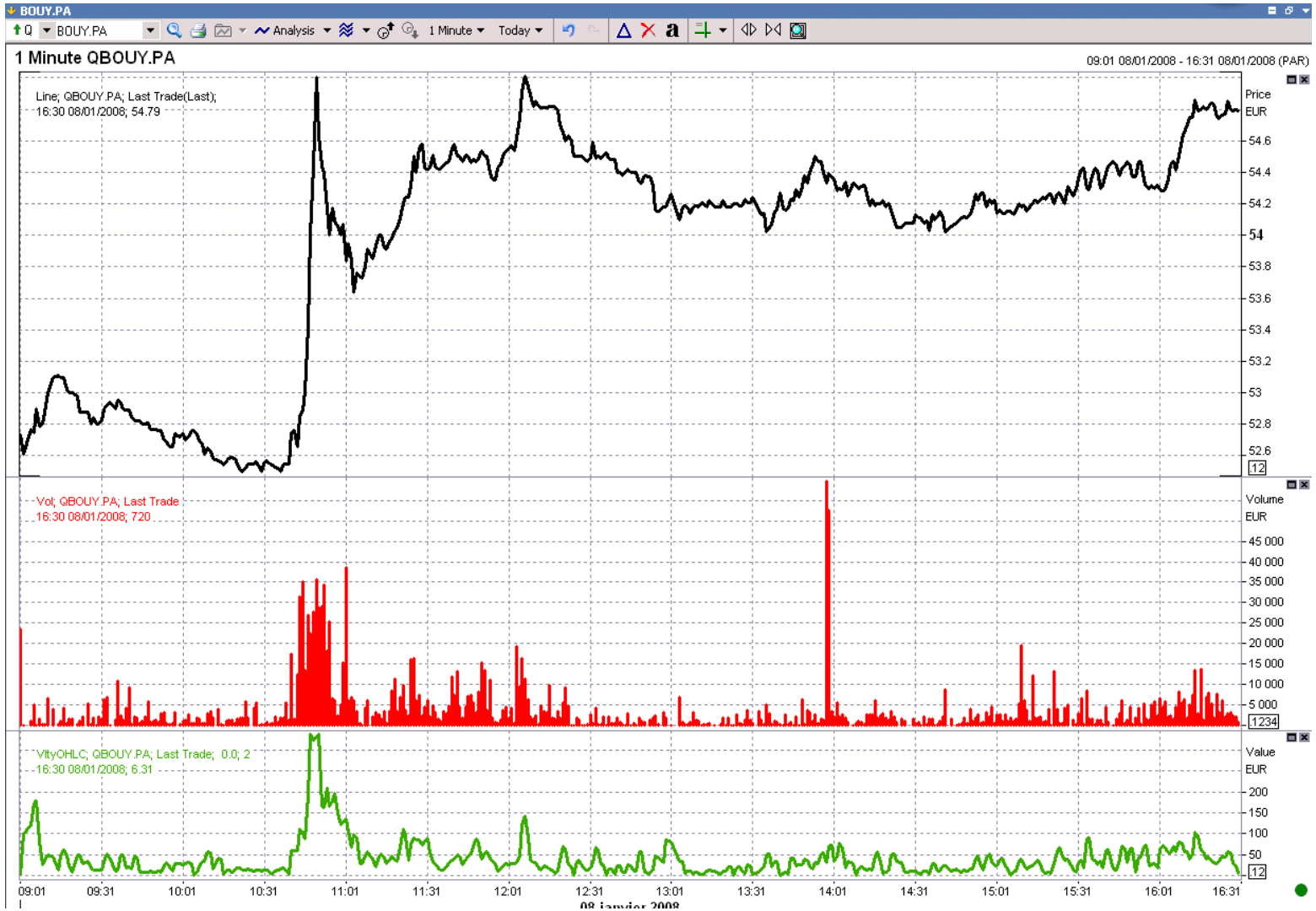
Theoretical developments in [LGR1O] (macroscopic scale) and [LLLL13] (microscopic scale) .

Example



TF1 price around the announcement of the end of advertising on public channels (8 Jan 2008).

Example



Bouygues price around the announcement of the end of advertising on public channels (8Jan. 2008).

Cost of Liquidity

What is liquidity ?

- How long you are have to wait until you trade the security at the fair price
- How much price rebate, you have to give in order to be able to trade simultaneously

From the viewpoint of an **investor** , cost to buy or sell a security is made of

- Fixed explicit costs (fees, memberships, technology, best exec proof investments, etc);
- Variable (implicit) costs, made of
  - **Bid-Ask spread** (and other liquidity measures [AMP1O]),
  - C**ost of an option** [Lon95],
  - **Market Impact** related to the capacity of his strategies [ATHLO5],
  - **Information leakage** related to adverse selection [GS8O].

Hence from the viewpoint of an **issuer**, having liquid shares lowers his cost of capital.

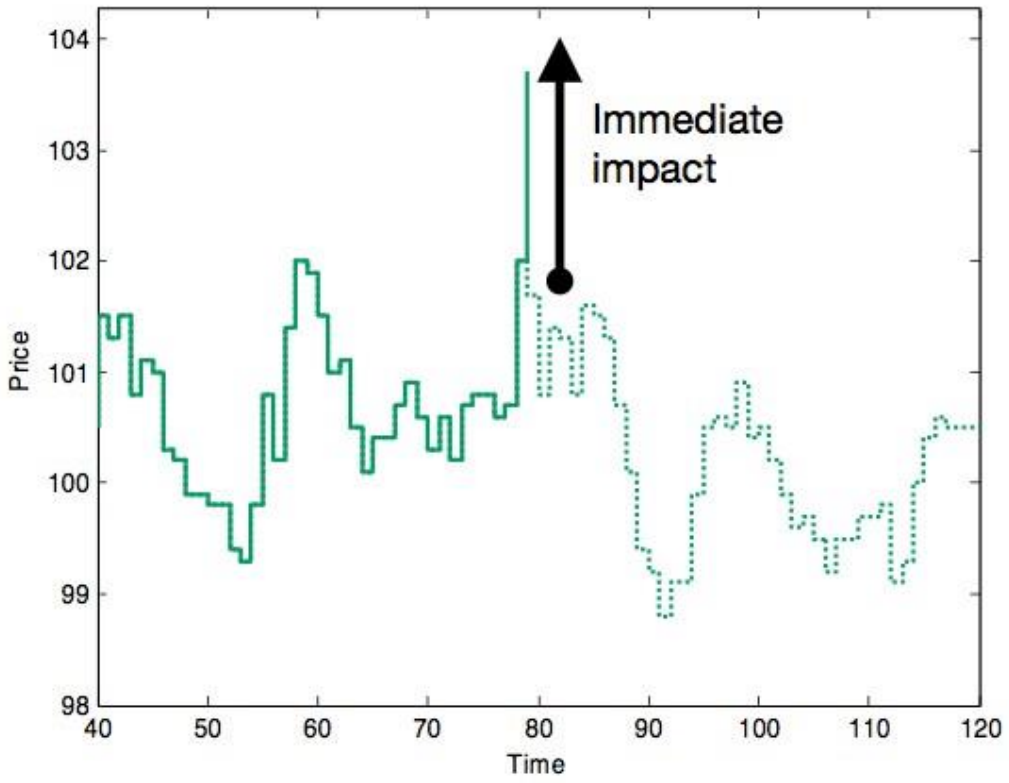
Market makers peculiar role

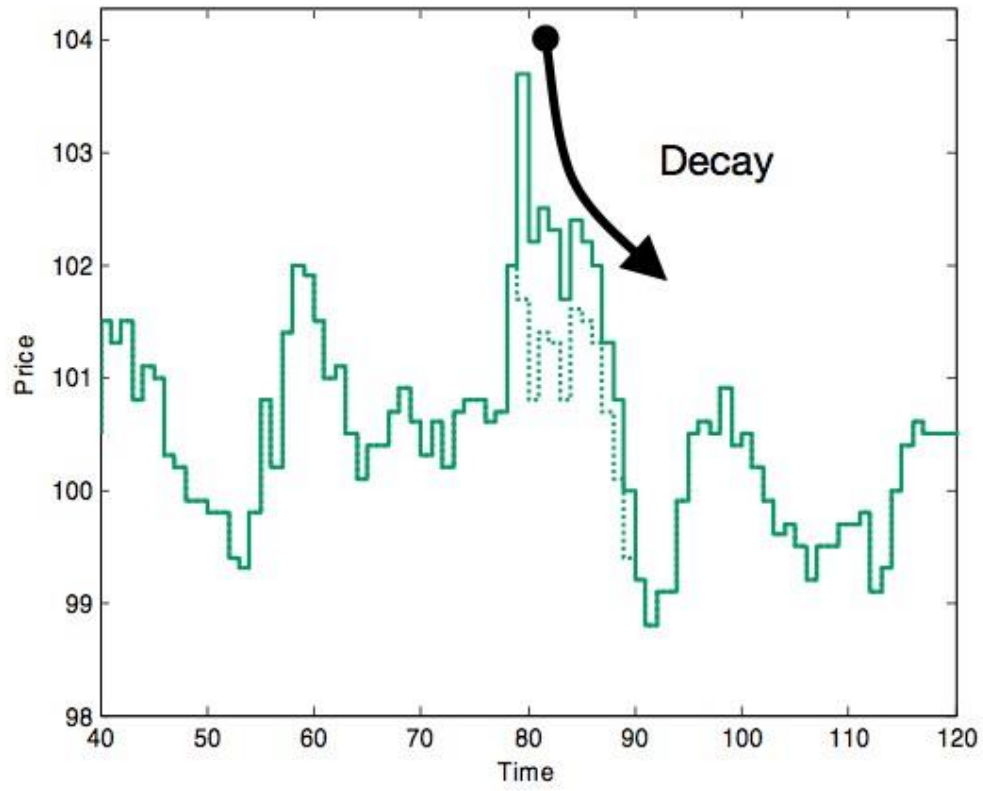
Market makers are said to increase liquidity [HS83] because they:

- ❑ Increase the number of trades successfully executed in the market
- ❑ While bearing the **inventory risk/costs**
- ❑ Reduce the bid-ask spread (i.e. linear transaction costs);
- ❑ Digest information while exposed to **adverse selection**

Adverse selection comes from information asymmetry between other investors and market makers. To protect themselves, market markers implemented as much **bilateral trading** as possible.

Market makers and market impact

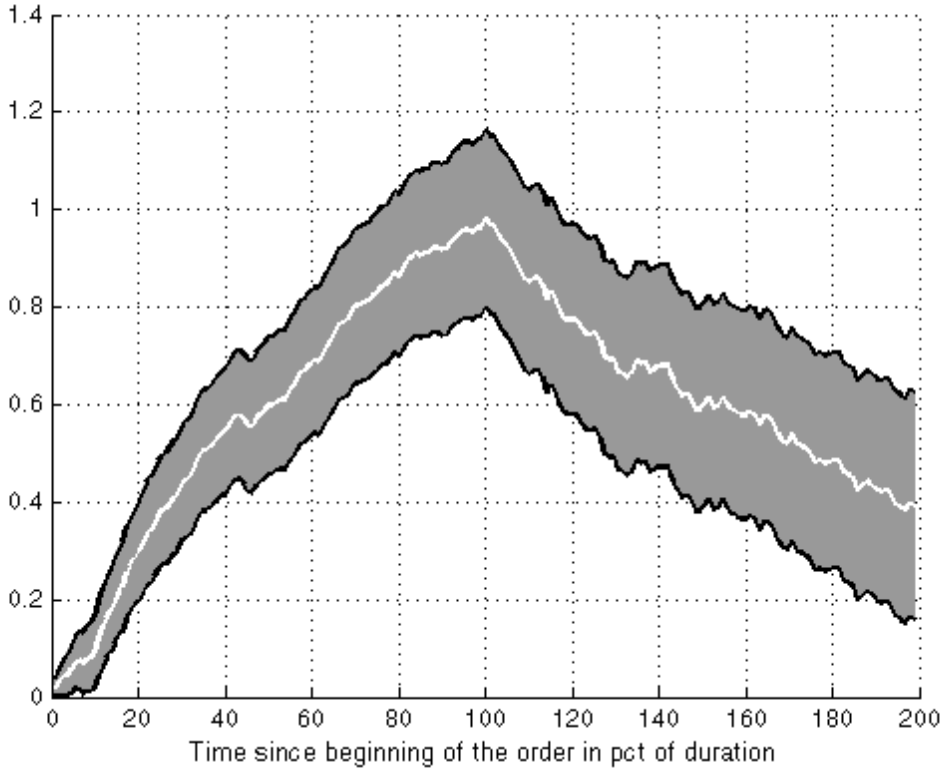




- market impact is the difference between the effective price dynamics (given some trades) and what it "would have been" without these trades;
- market makers should be able to "arbitrage" such moves, at the benefit of the price efficiency.

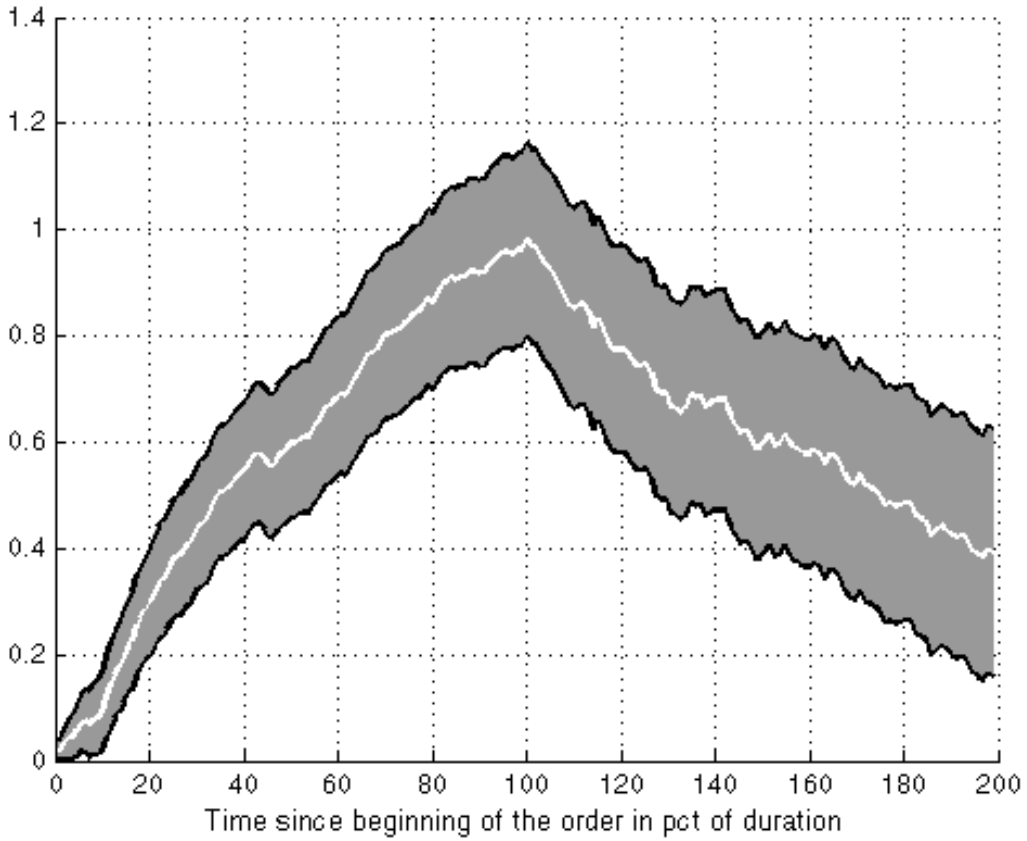
Market impact of large trades is a reality

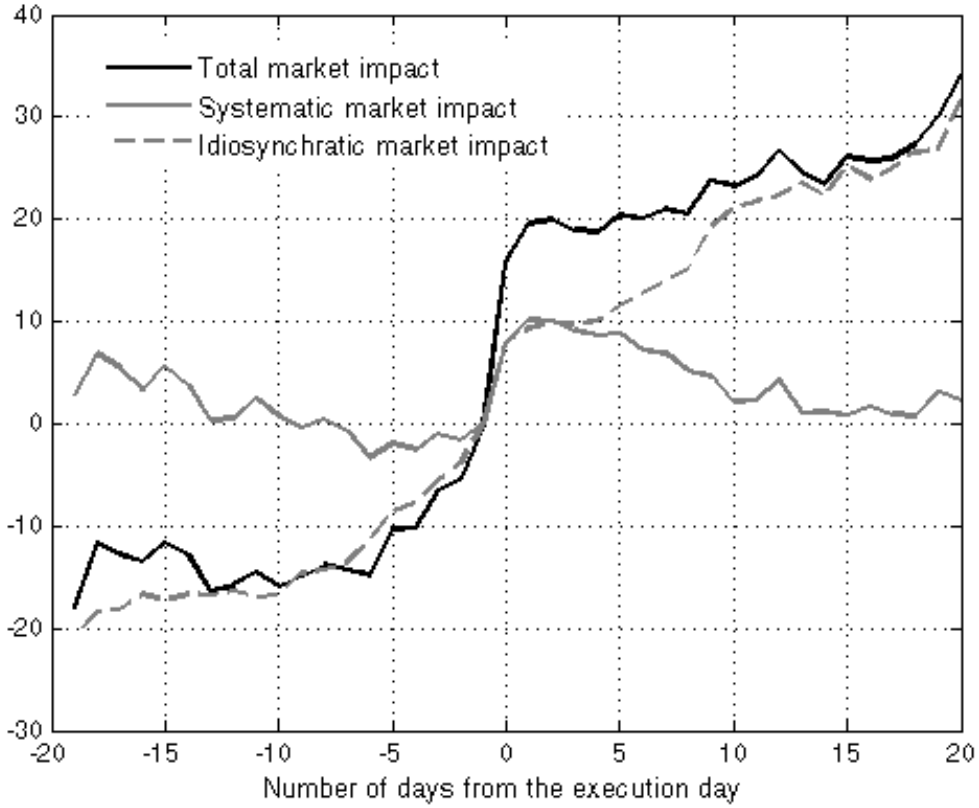
Real market impact of large orders from [LLB+13]



- Price move (in bid-ask spread, the end of the meta order is at $T = 100$)
- market impact has two phases: an immediate one (due to the volume pressure), and a decay

The meaning of permanent market impact



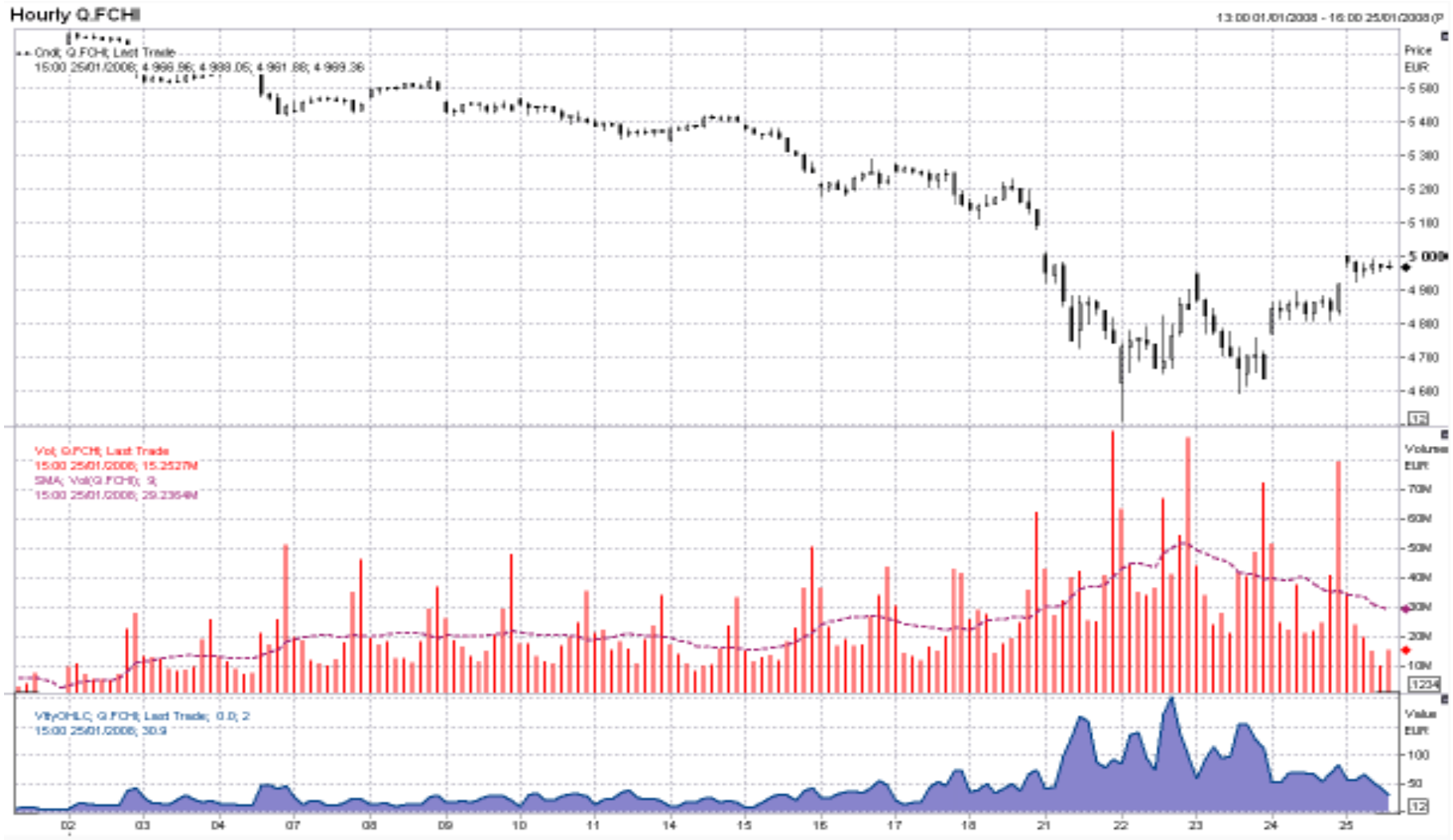


- Price move (in bid-ask spread, the end of the meta order is at $T = 100$)
- market impact has two phases: an immediate one (due to the volume pressure), and a decay

- Is **Permanent Market Impact** just realization of price discovery (backward)?
- Is **Market Impact** information leakage (backward) or mechanical pressure on the orderbook dynamics (forward)?

Extreme market impact events

Kerviel Market Impact - Studied in “Large bets and market Crashes” [KO13].



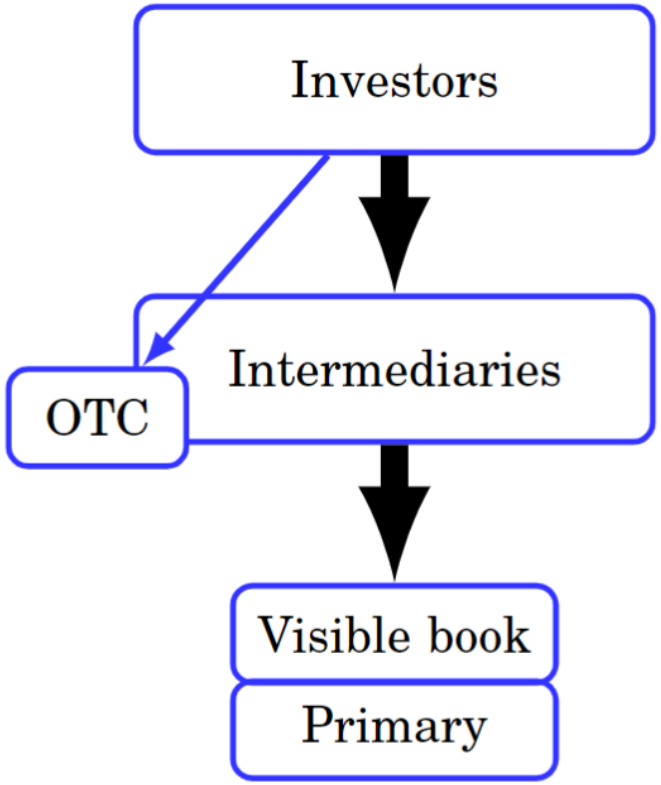
Trading in a Fragmented Market

In practice: the case of equity markets

MiFID in Europe and Reg NMS in the US opened the door to more competition between exchanges. Targeting:

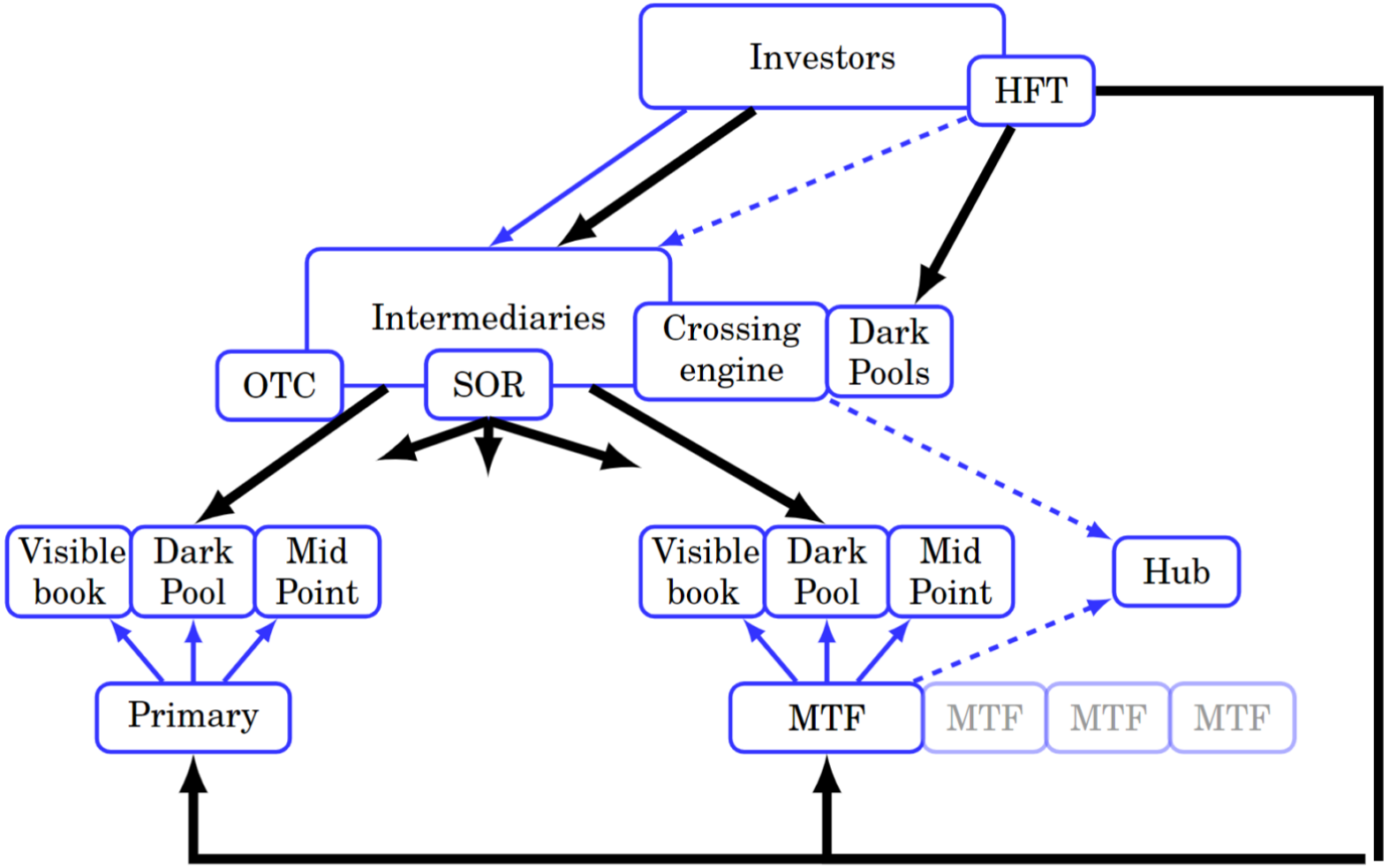
- increase the **quality of service** , **more (enough?) reliable and faster matching engines**
- better, more meaningful and easier to **access information** , **web sites of Chi-X BATS and fidessa**
- decrease **prices** , **lower trading fees, maker/taker fees**
- incentive to **innovation** . **Pegged orders, hidden orders, new matching rules**
- No significant change in **listing**.
- No competition on **fixing (close) auctions**.

The emergence of a new market structure



Trading before market fragmentation

The emergence of a new market structure



The trading now takes place on a distributed network of heterogenous trading platforms.

New participants

1) MTFs offered trading in a **one size fits all** approach,
2) they needed liquidity providers, thus offered rebate to limit orders,
3) statistical arbitrage technology (overnight risk) made automated market makers to produce **High Frequency Traders**.
4) As HFT provided competitive quotes, Brokers had to implement **Smart Order Router** (under regulatory pressure, since MiFID decided to implement a two-layered competition),
5) This sudden **increase of complexity** (and a lack of education) frightened some participants (suspecting that the profit of HFT were their lost),
6) Trading platforms offered **anonymity** features, claiming it will reduce information leakage (birth of European **Dark Pools**),
7) Exchanges provided **internalization** features, and brokers provided crossing capabilities,
8) High fixed costs and high competition leaded to **mergers** (LSE +Turquoise, BATS +Chi-X, ICE +Nyse, Euronext +?, etc);
9) Exchanges are close to be **technology vendors**.

A standard optimization paradox

- The more parameters you add to a situation, the **better the optimum**.
- But the **more complex to find**.

A standard optimization paradox

- The more parameters you add to a situation, the **better the optimum**.
- But the **more complex** to find.
- simultaneously, a large academic literature emerged to **optimize the trading process** (see [Leh13] in the *Handbook on systemic risk*, 2O13 for a review):
- as a result, a large trader can now liquidate a position using a majority of limit (liquidity adding) orders. Optimal schemes to seek liquidity on a large set of pools are known [PLL11] (even the market making problem is solved [GLFT13]).

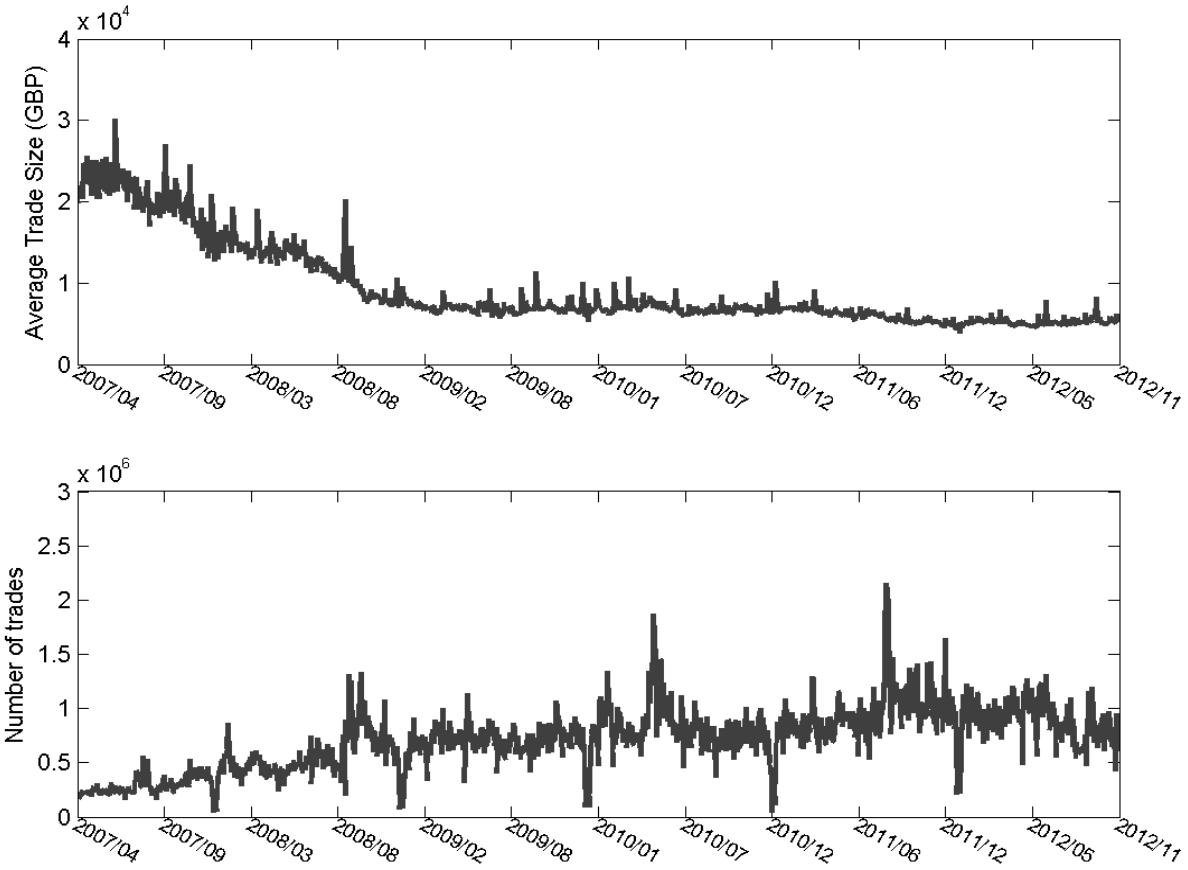
The notion of liquidity changed. A dynamical and probabilistic approach is now needed.

Orders or meta-orders?

A transaction occurs when two orders match, usually one is fully executed and the order only partially executed.

- the partially executed order has more informatio leakage (and potentially more impact) than the fu executed one;
- the smaller the orders, the better.
- **Electronic trading** is a way to shu½e liquidity at a scale.

Evolution of the Average Trade Size and the number trade per day on the LSE.



A typical fragmented stock

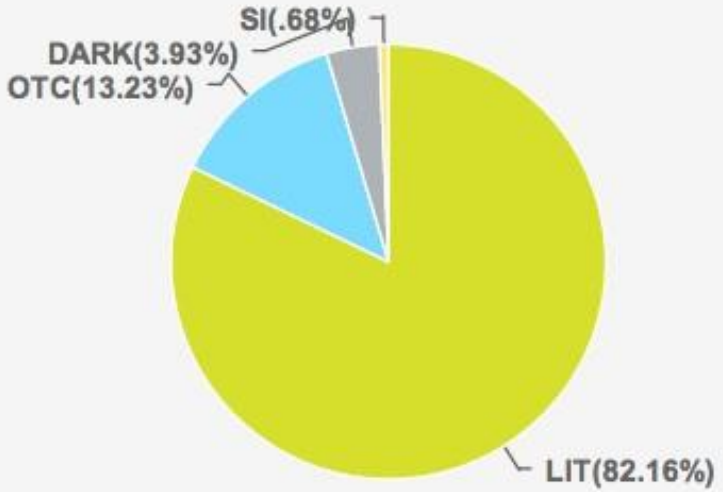
These numbers add all the transactions: they are good to estimate exchanges revenues.

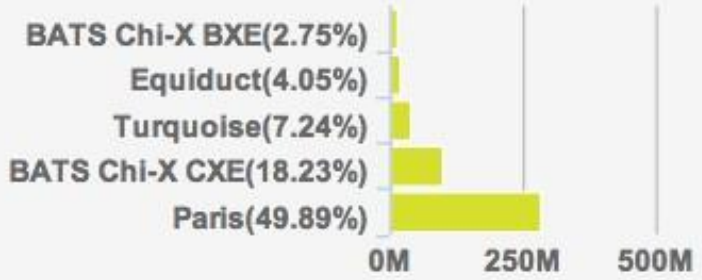
CREDIT AGRICOLE

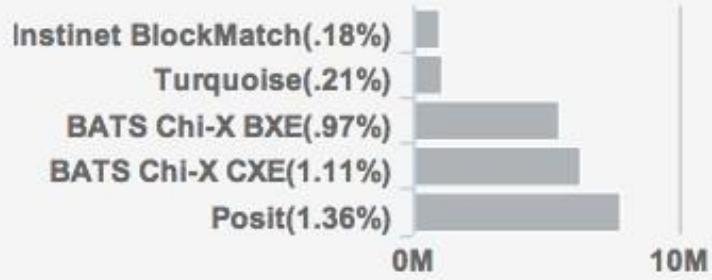
Monday, 2013 November 25 to Friday, 2013 November 29. View this in Fragulator Live

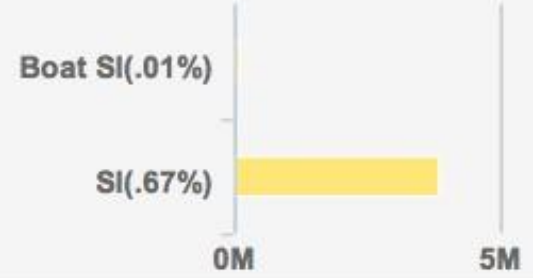
Market share

Lit venues	OTC reporting	Dark venues	Systematic internallisers
82.16%	13.23%	3.93%	.68%









A Different Way to Fragment in the US vs. in Europe

Regulations (Reg NMS and MiFID) implemented competition across trading venues: Exchange / Regulated Markets; ECN /MTF (Multilateral Trading Facilities).

In the US

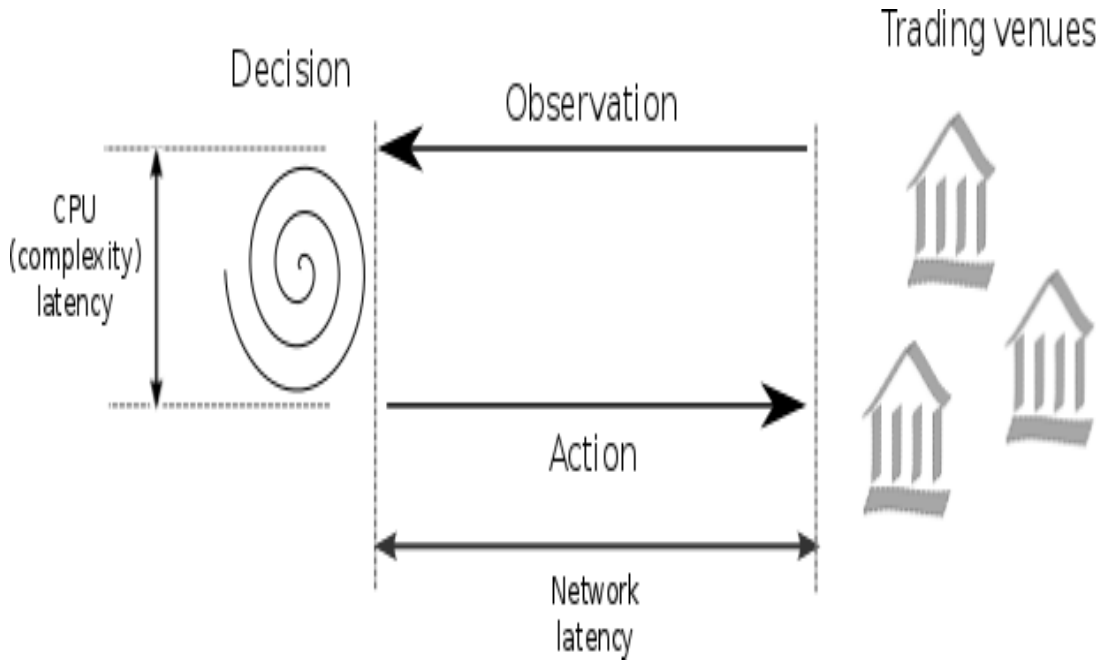
- The “trade through rule” demand to trading venues to re-route orders that can obtain a better price “somewhere”;
- That for, a “National Bid and Offer” (NBBO) is available to venues;
- 3 SIP (centralized Securities Information Processor) aggregate quotes coming from all venues to build a **Consolidated Tape**. They are maintained by the **The Consolidated Tape Association (CTA)**.
- They are then sent back to venues so that the NBBO is known by everyone.
- Thanks to this mechanism an order can be send “blindly” to any venue and take profit of competition (on the paper…).

Consolidated Tapes

In Europe, no similar mechanism have been implemented

- because when you compare two best bids (or asks) in the US, if one is larger than the other, you can be sure you will have more selling on this venue at the end of the day, even if you count the post trading costs (clearing and settlement costs mainly: there is only one silo).
- In Europe post trading costs can be different from one venue to the other (because of post trade costs),
- hence a consolidated tape will not tell a trader a price is more attractive on one venue or on the other.
- The Regulator demanded to broker to implement Smart Order Routers on their side.

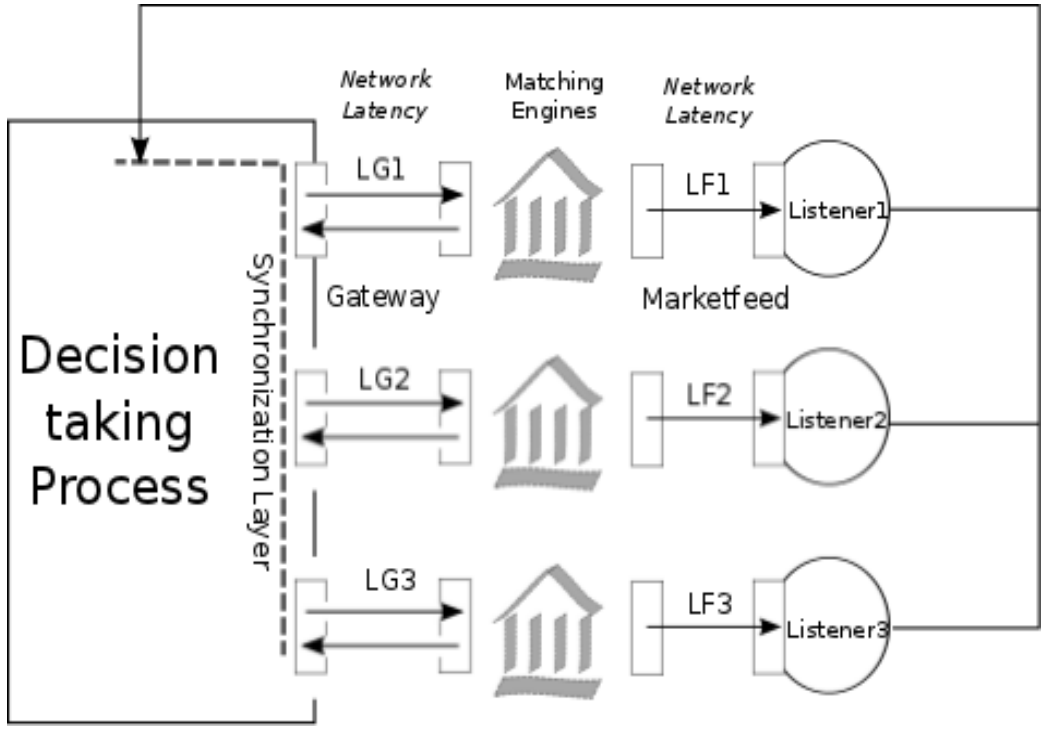
A SOR is parametrized to implement a “Best Execution Policy” (choice of the venues, priority /ranking rules, etc).



It is a little more complex…

The latency complexifies the story:

- If a SOR is at 1 ms from venue *A*, 2 ms from venue *B* and 3 ms from venue *C*,
- when he look at the three orderbooks, they are **not synchronized**,
- since the SOR has an SLE connection, it can have some insight about the accuracy of its snapshots.
- If the SOR see 2O shares at the offer at 1O.OO on *A*, 3O at the same price on *B* and 5O on *C*, and if the SOR want to buy 1OO shares, it can split it in 2O(*A*)+3O(*B*)+5O(*C*) and send the three orders (i.e. 3 insert IOC Buy orders at 1O.OO) at t_O
- but the 3 orders will reach the venues at $t_O + 1$, $t_O + 2$ and $t_O + 3$ ms (respectively).



If 2O of the 5O shares at *C* are duplicates of the 2O at *A* owned by a trader hosted at *A* and with a latency of 1 ms to *C*, he can cancel his shares before the SOR order reaches *C*.
In such a case the SOR will obtain 7O=2O(*A*)+2O(*B*)+3O(*C*) instead of the 1OO it expected

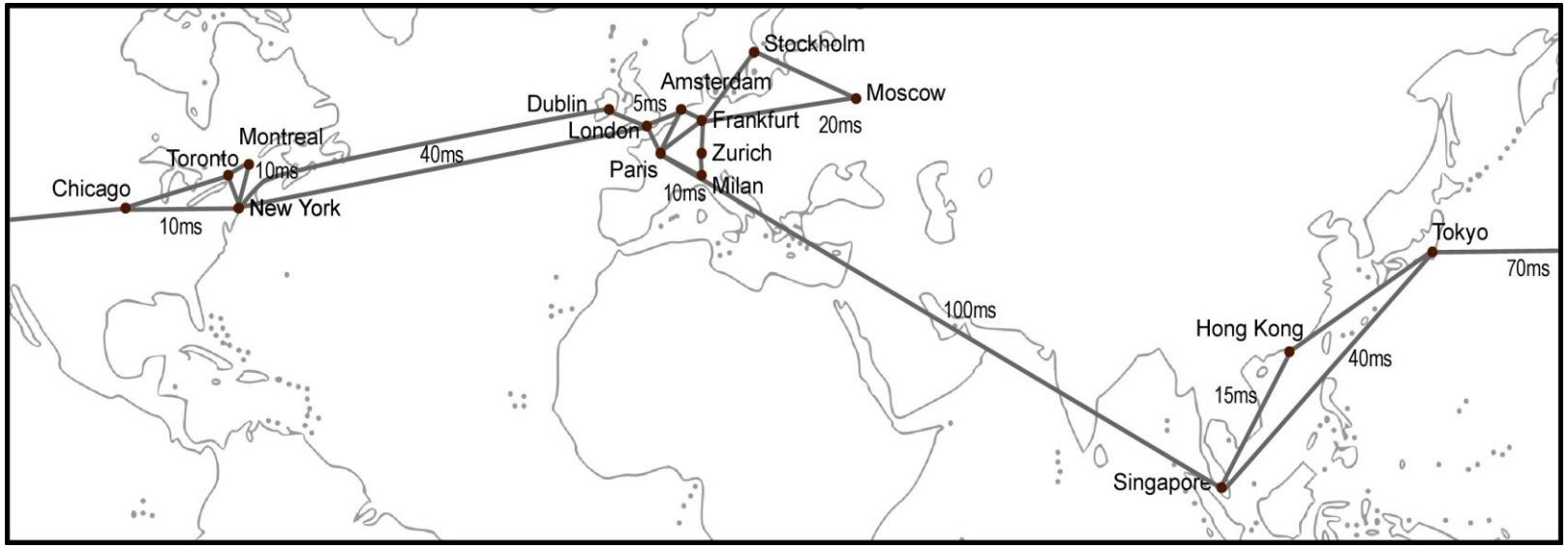
Dealing with latency

One solution is to delay the order to *A* by 2 ms and to *B* by 1 ms, so that the three orders reach the venues simultaneously. But the delays are not that deterministic, and waiting more means having a largest probability of an exogenous orderbook change.

Hardware can be useful.

- GPU is not that good since it is slow to transfer data from a motherboard to the GPU, and in trading we talk about being able to deal with a fast flow of data.
- The flow to work on is made of the SLC.
- FPGA is better since it can embed physical IP layers, hence you can put on an FPGA: (1) the reading of the flow, (2) its enrichment with indicators. Without paying that more latency.

High Frequency or Low Latency?



Low latency means to be able to react first to external information. It is not mandatory for this external information to be frequent. To implement such strategies you must be fast on long distances.

High frequency means to be able to (re)adjust your orders 1OO times per second. To implement this you need to be very close to the orderbooks you are working in.

You cannot be simultaneously close to 2 venues that are not located in the same datacenter.

Timestamping and Event Ordering

Remark on Timestamping and Event Ordering:

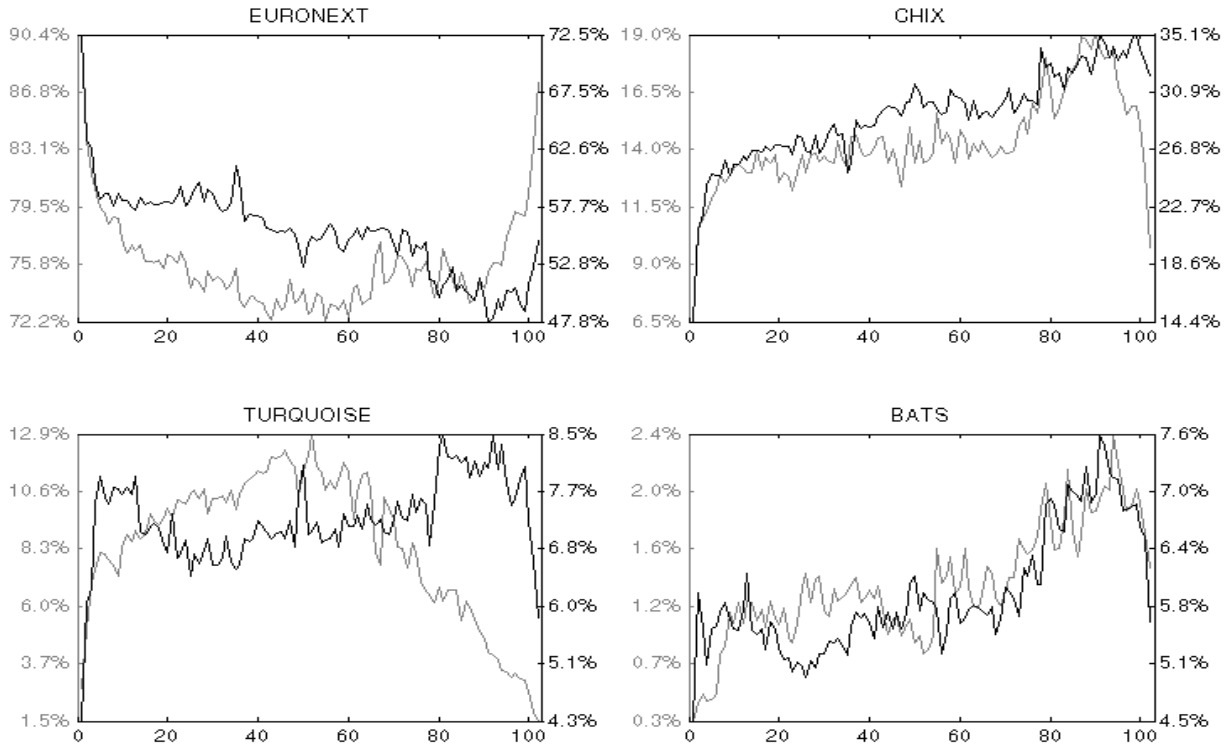
- Two events occurring simultaneously at points *A* and *B* in space are perceived in a different orders with respect to the position of the observer.
- When an observer suffers from a delay, he can choose between (1) taking decisions on estimators of the state of the orderbook, or (2) implementing a simultaneous optimization on its strategy.

As an example, **to trade in Dark Pools** :

- [GNKV1O] estimates the liquidity in each pool and implements a deterministic optimization;
- [ABD1O] uses a minimum regret approach;
- [PLL11] implements a stochastic implementation of an optimal trading scheme.

The first approach is good for on opportunistic trading (hedge fund), the second for a rare and not really flexible flow (investor), the last one is good for very large systematic flow (broker).

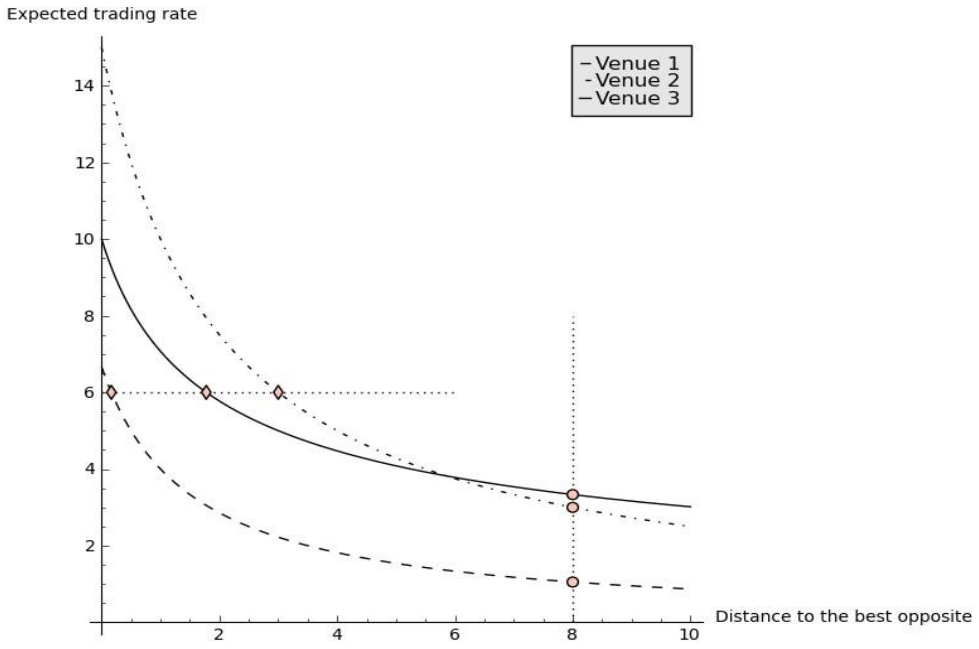
Measuring fragmentation (examples)



Market Share of venues on a typical Fench Stock in 2012

- The probability that a trade typically occurred on a given venue should take into account the time,
- the probability to find liquidity on a venue? (given a trade occurred)
- The Average Trade Size on each venue (is in fact the average size of a limit order..).
- Cost of a “roundtrip” for a given amount of Euros on each venue.

Viewpoint of an optimal (trader) router



Given that you know in advance the flow Φ_k of liquidity consumption on each venue *k*, you can choose any combination (to focus on price or speed).

The estimation of the flow is difficult, since the picture you see is outdated at the rhythm of the updates of the orderbook: if the updating rate ρ is 250Hz, then the time to react is

$$\tau_{reaction} = \frac{1}{2\rho} = 2ms$$

In [PLL11] and [LLP13], we show how to adjust the limit price and the quantities to send to each venue for an arbitrary criterion (in a stochastic world). And we provide uncertainty bounds on the result.

Execution and Algorithmic Trading

Algorithmic trading

- In most brokerage firms, Algorithmic trading executes more than 80% of the orders (in value). It is thus used internally (to execute care or corporate orders) and by (low touch) clients.
- Algo users pay fees (2 to 8 bps) and have no guarantee on the execution price.
- Buy sides would like to have Algos the most similar as possible from one provider to another, since integration in the trading tools is a never-ending challenge. But Sell sides try to differentiate, leading to Algos with fancy names and "new" features.

Algo types

- **Long duration Algos** are controlled via a *Benchmark*, giving an indication on the criterion to minimize (reflecting user's belief in good proxies for liquidity and risk), and having different parameters.
- **Short duration Algos** (or Robots) are synthesizing smart features of orders (icebergs, on the close, stop orders, snipers, etc) or seek liquidity on more than one platform (SOR, liquidity seekers). Their parameters make the balance between speed and price.
- Algos can be **customized**: benchmarked algos can send robots, and the type of an algo can **change on market conditions**, like: "*start with a VWAP and switch to PoV if the market volume is more than 2 times the ADV*".
- **Program trading Algos** are dedicated to portfolio trading, with synchronization features (in cash or market exposure).

Trading algorithms: Typical uses

Benchmark	Region of preference	Order characteristics	Market context	Type of hedged risk
PoV	Asia	Large order size (more than 1O% of ADV: Average daily consolidated volume)	Possible negative news	Do not miss the rapid propagation of an unexpected news event (especially if I have the information)
VWAP / TWAP	Asia and Europe	Medium size (from 5 to 15% of ADV)	Any “unusual” volume is negligi- ble	Do not miss the slow propagation of information in the market
Implementation Shortfall (IS)	Europe and US	Small size (O to 6% of ADV)	Possible price opportunities	Do not miss an unexpected price move in the stock
Liquidity Seeker	US (Europe)	Any size	The stock is expected to “oscil- late” around its “fair value”	Do not miss a liquidity burst or a relative price move on the stock

Trading algorithms: Typical features

Benchmark	Type of stock	Type of trade	Main feature
PoV	Medium to large market depth	(1) Long duration position	(1) Follows current market flow, (2) Very reactive, can be very aggressive, (3) More price opportunity driven if the range between the max percent and min percent is large
VWAP / TWAP	Any market depth	(1) Hedging order, (2) Long duration position, (3) Unwind tracking error (delta hedging of a fast evolving inventory)	(1) Follows the "usual" market flow, (2) Good if market moves with unexpected volumes in the same direction as the order (up for a buy order), (3) Can be passive
Implementation Shortfall (IS)	Medium liquidity depth	(1)Alpha extraction, (2) Hedge of a non-linear position (typically Gamma hedging), (3) Inventory- driven trade	(1) Will finish very fast if the price is good and enough liquidity is available, (2) Will "cut losses" if the price goes too far away
Liquidity Seeker	Poor fragmented market depth	(1) Alpha extraction, (2) Opportunistic position mounting, (3) Already split / scheduled order	(1) Relative price oriented (from one liquidity pool to another, or from one security to another), (2) Capture liquidity everywhere, (3) Stealth (mini-mum information leakage using fragmentation)

Pre and Post trade analysis, TCA consultancy

With electronic trading, a full range of services is offered:

Buy sides are performing TCA (Transaction Cost Analysis) to compare their brokers on a monthly basis

Pre trade analysis provides automated assistance to Algo parameters tuning (taking market impact / market risk into account), it computes betas, exposures, etc of a list of algos sent by Dealing Desks.

Post trade analysis: at the end of the day, the generation of a report is automated to allow to understand fast what happened during the day for each Dealing desk. Performances are compared with unexpected and expected new arrivals, and with usual performances and performances of other clients.

Monitoring: is conducted during the life of algos, analytics are grouping trading lines according to market conditions to dynamically explain the potential causes of performance degradation, so that sales-traders can adjust some parameters or call the Buy side Dealing Desk

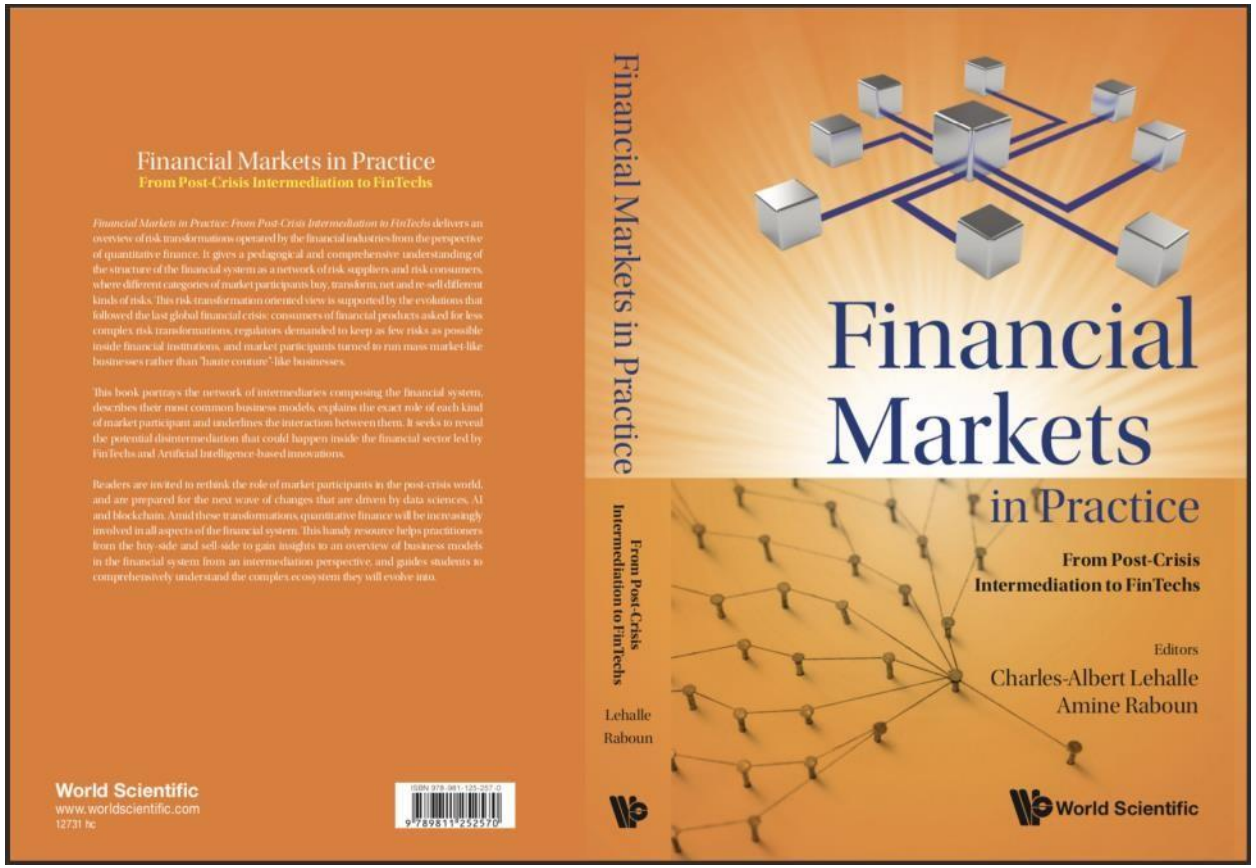
Execution Consultancy is available on demand too. Expersts discuss with clients to help them to tune parameters for their own use, or to design customize new Alogs. Bbacktest can be done, models or parameters can be sent to clients by FIX/FTP

While it is easy to un-plug an Algo provider, brokers try to develop stickiness of clients “vous ne viendrez plus chez nous au hazard”

Conclusion

Questions ?

To go further



To go further:

Financial Markets in Practice: From Post Crisis Intermediation to FinTechs
by Charles-Albert Lehalle
and Amine Raboun

(World Scientific Publisher, June 2022)

References

- Alekh Agarwal, Peter L. Bartlett, and Max Dama, *Optimal Allocation Strategies for the Dark Pool Problem*, Proceedings of The Thirteenth International Conference on Artificial Intelligence and Statistics (AISTATS) (Y. W. Teh and M. Titterington, eds.), vol. 9, May 2010, pp. 9–16.
- Yakov Amihud, Haim Mendelson, and Lasse H. Pedersen, *Liquidity and Asset Prices*, Social Science Research Network Working Paper Series (2010).
- Robert Almgren, Chee Thum, Emmanuel Hauptmann, and Hong Li, *Direct Estimation of Equity Market Impact*, Risk **18** (2005), 57–62.
- Antoine Bouveret, Cyrille Guillaumie, Carlos A. Roqueiro, Christian Winkler, and Steffen Nauhaus, *High-frequency trading activity in EU equity markets*, Tech. report, ESMA, 2014.
- Nataliya Bershova and Dmitry Rakhlin, *The Non-Linear Market Impact of Large Trades: Evidence from Buy-Side Order Flow*, Tech. report, December 2012.
- Olivier Guéant, Charles-Albert Lehalle, and Joaquin Fernandez-Tapia, *Dealing with the inventory risk: a solution to the market making problem*, Mathematics and Financial Economics **4** (2013), no. 7, 477–507.

References

- Kuzman Ganchev, Yuriy Nevmyvaka, Michael Kearns, and Jennifer W. Vaughan, *Censored exploration and the dark pool problem*, Commun. ACM **53** (2010), no. 5, 99–107.
- Sanford J. Grossman and Joseph E. Stiglitz, *On the Impossibility of Informationally Efficient Markets*, The American Economic Review **70** (1980), no. 3, 393–408.
- Tsy Ho and H. R. Stoll, *The dynamics of dealer markets under competition*, Journal of Finance **38** (1983), no. 4, 1053–1074.

- Albert P. Kyle and Anna A. Obizhaeva, *Large Bets and Stock Market Crashes*, Tech. report, Market Microstructure: Confronting Many Viewpoints, July 2013.
- Charles-Albert Lehalle, *Market Microstructure knowledge needed to control an intra-day trading process*, Handbook on Systemic Risk (Jean-Pierre Fouque and Joseph Langsam, eds.), Cambridge University Press, May 2013.
- Charles-Albert Lehalle, Olivier Guéant, and Julien Razafinimanana, *High Frequency Simulations of an Order Book: a Two-Scales Approach*, Econophysics of Order-Driven Markets (F. Abergel, B. K. Chakrabarti, A. Chakraborti, and M. Mitra, eds.), New Economic Windows, Springer, 2010.

References

- Charles-Albert Lehalle, Sophie Laruelle, Romain Burgot, Stéphanie Pelin, and Matthieu Lasnier, *Market Microstructure in Practice*, World Scientific publishing, 2013.
- Aimé Lachapelle, Jean-Michel Lasry, Charles-Albert Lehalle, and Pierre-Louis Lions, *Efficiency of the Price Formation Process in Presence of High Frequency Participants: a Mean Field Game analysis*, May 2013.
- Sophie Laruelle, Charles-Albert Lehalle, and Gilles Pagès, *Optimal posting price of limit orders: learning by trading*, Mathematics and Financial Economics **7** (2013), no. 3, 359–403.

- Francis A. Longstaff, *How Much Can Marketability Affect Security Values?*, The Journal of Finance **50** (1995), no. 5, 1767–1774.
- Robert C. Merton, *A Functional Perspective of Financial Intermediation*, Financial Management **24** (1995), no. 2, 23+.
- Esteban Moro, Javier Vicente, Luis G. Moyano, Austin Gerig, Doyne J. Farmer, Gabriella Vaglica, Fabrizio Lillo, and Rosario N. Mantegna, *Market impact and trading profile of large trading orders in stock markets*, August 2009.

References


Gilles Pagès, Sophie Laruelle, and Charles-Albert Lehalle, *Optimal split of orders across liquidity pools: a stochastic algorithm approach*, SIAM Journal on Financial Mathematics **2** (2011), 1042-1076.